

# Boundedness for fractional Hardy-type operator on variable exponent Herz-Morrey spaces

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## Abstract

In this paper, the fractional Hardy-type operator of variable order  $\beta(x)$  is shown to be bounded from the variable exponent Herz-Morrey spaces  $M\dot{K}_{p_1, q_1(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)$  into the weighted space  $M\dot{K}_{p_2, q_2(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n, \omega)$ , where  $\alpha(x) \in L^\infty(\mathbb{R}^n)$  be log-Hölder continuous both at the origin and at infinity,  $\omega = (1 + |x|)^{-\gamma(x)}$  with some  $\gamma(x) > 0$  and  $1/q_1(x) - 1/q_2(x) = \beta(x)/n$  when  $q_1(x)$  is not necessarily constant at infinity.

## 1 Introduction

Let  $f$  be a locally integrable function on  $\mathbb{R}^n$ . The  $n$ -dimensional Hardy operator is defined by

$$\mathcal{H}(f)(x) := \frac{1}{|x|^n} \int_{|t| < |x|} f(t) dt, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

In 1995, Christ and Grafakos<sup>[1]</sup> obtained the result for the boundedness of  $\mathcal{H}$  on  $L^p(\mathbb{R}^n)$  ( $1 < p < \infty$ ) spaces, and they also found the exact operator norms of  $\mathcal{H}$  on this space. In 2007, Fu et al<sup>[2]</sup> gave the central BMO estimates for commutators of  $n$ -dimensional fractional and Hardy operators. And recently, the first author<sup>[3–9]</sup> also considers the boundedness for Hardy operator and its commutator in (variable exponent) Herz-Morrey spaces.

Nowadays there is an evident increase of investigations related to both the theory of the variable exponent function spaces and the operator theory in these spaces. This is caused by possible applications to models with non-standard local growth (in elasticity theory, fluid mechanics, differential equations and image processing, see for example [10–14] and references therein) and is based on the breakthrough result on boundedness of the Hardy-Littlewood maximal operator in these spaces (for more details see [15–25] et al).

We first define the  $n$ -dimensional fractional Hardy-type operators with variable order  $\beta(x)$  as follows.

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**Definition 1.1** Let  $f$  be a locally integrable function on  $\mathbb{R}^n$ ,  $0 \leq \beta(x) < n$ . The  $n$ -dimensional fractional Hardy-type operators of variable order  $\beta(x)$  are defined by

$$\mathcal{H}_{\beta(\cdot)}(f)(x) := \frac{1}{|x|^{n-\beta(x)}} \int_{|t| < |x|} f(t) dt, \quad (1.0a)$$

$$\mathcal{H}_{\beta(\cdot)}^*(f)(x) := \int_{|t| \geq |x|} \frac{f(t)}{|t|^{n-\beta(x)}} dt, \quad (1.0b)$$

where  $x \in \mathbb{R}^n \setminus \{0\}$ .

Obviously, when  $\beta(x) = 0$ ,  $\mathcal{H}_{\beta(\cdot)}$  is just  $\mathcal{H}$ , and denote by  $\mathcal{H}^* := \mathcal{H}_{\beta(\cdot)}^* = \mathcal{H}_0^*$ . And when  $\beta(x)$  is constant,  $\mathcal{H}_{\beta(\cdot)}$  and  $\mathcal{H}_{\beta(\cdot)}^*$  will become  $\mathcal{H}_\beta$  and  $\mathcal{H}_\beta^*$  respectively.

The Riesz-type potential operator with variable order  $\beta(x)$  is defined by

$$I_{\beta(\cdot)}(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta(x)}} dy, \quad 0 < \beta(x) < n. \quad (1.1)$$

In 2004, Diening<sup>[26]</sup> proved Sobolev's theorem for the potential  $I_\beta$  on the whole space  $\mathbb{R}^n$  assuming that  $p(x)$  is constant at infinity ( $p(x)$  is always constant outside some large ball) and satisfies the same logarithmic condition as in [27]. Another progress for unbounded domains is the result of Cruz-Uribe et al<sup>[18]</sup> on the boundedness of the maximal operator in unbounded domains for exponents  $p(x)$  satisfying the logarithmic smoothness condition both locally and at infinity.

In [28], Kokilashvili and Samko prove Sobolev-type theorem for the potential  $I_{\beta(\cdot)}$  from the space  $L^{p(\cdot)}(\mathbb{R}^n)$  into the weighted space  $L_\omega^{q(\cdot)}(\mathbb{R}^n)$  with the power weight  $\omega$  fixed to infinity, under the logarithmic condition for  $p(x)$  satisfied locally and at infinity, not supposing that  $p(x)$  is constant at infinity but assuming that  $p(x)$  takes its minimal value at infinity.

In addition, the theory of function spaces with variable exponent has rapidly made progress in the past twenty years since some elementary properties were established by Kováčik-Rákosník<sup>[15]</sup>.

In 2012, Almeida and Drihem<sup>[29]</sup> discuss the boundedness of a wide class of sublinear operators on Herz spaces  $K_{q(\cdot)}^{\alpha(\cdot), p}(\mathbb{R}^n)$  and  $\dot{K}_{q(\cdot)}^{\alpha(\cdot), p}(\mathbb{R}^n)$  with variable exponent  $\alpha(\cdot)$  and  $q(\cdot)$ . Meanwhile, they also established Hardy-Littlewood-Sobolev theorems for fractional integrals on variable Herz spaces. In 2013, Samko<sup>[30, 31]</sup> introduced a new Herz type function spaces with variable exponent, where all the three parameters are variable, and proved the boundedness of some sublinear operators (also ref. [32]). In 2014, Izuki and Noi<sup>[33]</sup> concerned with duality and reflexivity of Herz spaces  $K_{q(\cdot)}^{\alpha(\cdot), p(\cdot)}(\mathbb{R}^n)$  and  $\dot{K}_{q(\cdot)}^{\alpha(\cdot), p(\cdot)}(\mathbb{R}^n)$  with variable exponents. Moreover, in recently, Wu<sup>[9]</sup> considers the boundedness for fractional Hardy-type operator on Herz-Morrey spaces  $M\dot{K}_{p, q(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)$  with variable exponent  $q(\cdot)$  but fixed  $\alpha \in \mathbb{R}$  and  $p \in (0, \infty)$ .

Motivated by the above results, we are to investigate mapping properties of the fractional Hardy-type operators  $\mathcal{H}_{\beta(\cdot)}$  and  $\mathcal{H}_{\beta(\cdot)}^*$  within the framework of the variable exponent Herz-Morrey spaces  $M\dot{K}_{p, q(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)$ .

Throughout this paper, we will denote by  $|S|$  the Lebesgue measure and by  $\chi_S$  the characteristic function for a measurable set  $S \subset \mathbb{R}^n$ ;  $B(x, r)$  is the ball centered at  $x$  and of radius

$r$ ;  $B_0 = B(0, 1)$ .  $C$  denotes a constant that is independent of the main parameters involved but whose value may differ from line to line. For any index  $1 < q(x) < \infty$ , we denote by  $q'(x)$  its conjugate index, namely,  $q'(x) = \frac{q(x)}{q(x)-1}$ . For  $A \sim D$ , we mean that there is a constant  $C > 0$  such that  $C^{-1}D \leq A \leq CD$ .

## 2 Preliminaries

In this section, we give the definition of Lebesgue and Herz-Morrey spaces with variable exponent, and give basic properties and useful lemmas.

### 2.1 Function spaces with variable exponent

Let  $\Omega$  be a measurable set in  $\mathbb{R}^n$  with  $|\Omega| > 0$ . We first define variable exponent Lebesgue spaces.

**Definition 2.1** Let  $q(\cdot) : \Omega \rightarrow [1, \infty)$  be a measurable function.

(I) The Lebesgue spaces with variable exponent  $L^{q(\cdot)}(\Omega)$  is defined by

$$L^{q(\cdot)}(\Omega) = \{f \text{ is measurable function} : F_q(f/\eta) < \infty \text{ for some constant } \eta > 0\},$$

where  $F_q(f) := \int_{\Omega} |f(x)|^{q(x)} dx$ . The Lebesgue space  $L^{q(\cdot)}(\Omega)$  is a Banach function space with respect to the norm

$$\|f\|_{L^{q(\cdot)}(\Omega)} = \inf \left\{ \eta > 0 : F_q(f/\eta) = \int_{\Omega} \left( \frac{|f(x)|}{\eta} \right)^{q(x)} dx \leq 1 \right\}.$$

(II) The space  $L_{\text{loc}}^{q(\cdot)}(\Omega)$  is defined by

$$L_{\text{loc}}^{q(\cdot)}(\Omega) = \{f \text{ is measurable} : f \in L^{q(\cdot)}(\Omega_0) \text{ for all compact subsets } \Omega_0 \subset \Omega\}.$$

(III) The weighted Lebesgue space  $L_{\omega}^{q(\cdot)}(\Omega)$  is defined by as the set of all measurable functions for which

$$\|f\|_{L_{\omega}^{q(\cdot)}(\Omega)} = \|\omega f\|_{L^{q(\cdot)}(\Omega)} < \infty.$$

Next we define some classes of variable exponent functions. Given a function  $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ , the Hardy-Littlewood maximal operator  $M$  is defined by

$$Mf(x) = \sup_{r>0} r^{-n} \int_{B(x,r)} |f(y)| dy,$$

where  $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ .

**Definition 2.2** Given a measurable function  $q(\cdot)$  defined on  $\mathbb{R}^n$ , we write

$$q_- := \text{ess inf}_{x \in \mathbb{R}^n} q(x), \quad q_+ := \text{ess sup}_{x \in \mathbb{R}^n} q(x).$$

$$(I) \quad q'_- = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} q'(x) = \frac{q_+}{q_+ - 1}, \quad q'_+ = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} q'(x) = \frac{q_-}{q_- - 1}.$$

(II) Denote by  $\mathcal{P}(\mathbb{R}^n)$  the set of all measurable functions  $q(\cdot) : \mathbb{R}^n \rightarrow (1, \infty)$  such that

$$1 < q_- \leq q(x) \leq q_+ < \infty, \quad x \in \mathbb{R}^n.$$

(III) The set  $\mathcal{B}(\mathbb{R}^n)$  consists of all measurable functions  $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  satisfying that the Hardy-Littlewood maximal operator  $M$  is bounded on  $L^{q(\cdot)}(\mathbb{R}^n)$ .

**Definition 2.3** Let  $\alpha(\cdot)$  be a real-valued function on  $\mathbb{R}^n$ .

(I) The set  $\mathcal{C}_{loc}^{\log}(\mathbb{R}^n)$  consists of all local log-Hölder continuous functions  $\alpha(\cdot)$  satisfies

$$|\alpha(x) - \alpha(y)| \leq \frac{-C}{\ln(|x - y|)}, \quad |x - y| \leq 1/2, \quad x, y \in \mathbb{R}^n.$$

(II) The set  $\mathcal{C}_0^{\log}(\mathbb{R}^n)$  consists of all log-Hölder continuous functions  $\alpha(\cdot)$  at origin satisfies

$$|\alpha(x) - \alpha(0)| \leq \frac{C}{\ln(e + \frac{1}{|x|})}, \quad x \in \mathbb{R}^n. \quad (2.1)$$

(III) The set  $\mathcal{C}_\infty^{\log}(\mathbb{R}^n)$  consists of all log-Hölder continuous functions  $\alpha(\cdot)$  at infinity satisfies

$$|\alpha(x) - \alpha_\infty| \leq \frac{C_\infty}{\ln(e + |x|)}, \quad x \in \mathbb{R}^n, \quad (2.2)$$

where  $\alpha_\infty = \lim_{|x| \rightarrow \infty} \alpha(x)$ .

(IV) Denote by  $\mathcal{C}^{\log}(\mathbb{R}^n) := \mathcal{C}_{loc}^{\log}(\mathbb{R}^n) \cap \mathcal{C}_\infty^{\log}(\mathbb{R}^n)$  the set of all global log-Hölder continuous functions  $\alpha(\cdot)$ .

**Remark 1** The  $\mathcal{C}_\infty^{\log}(\mathbb{R}^n)$  condition is equivalent to the uniform continuity condition

$$|q(x) - q(y)| \leq \frac{C}{\ln(e + |x|)}, \quad |y| \geq |x|, \quad x, y \in \mathbb{R}^n.$$

The  $\mathcal{C}_\infty^{\log}(\mathbb{R}^n)$  condition was originally defined in this form in [18].

Next we define variable exponent Herz-Morrey spaces  $M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$ . Let  $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ ,  $A_k = B_k \setminus B_{k-1}$  and  $\chi_k = \chi_{A_k}$  for  $k \in \mathbb{Z}$ .

**Definition 2.4** Suppose that  $0 \leq \lambda < \infty$ ,  $0 < p < \infty$ ,  $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  and  $\alpha(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ . The variable exponent Herz-Morrey spaces  $M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$  is defined by

$$M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n) = \left\{ f \in L_{loc}^{q(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left( \sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f \chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}.$$

Compare the variable Herz-Morrey space  $M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$  with the variable Herz space<sup>[29]</sup>  $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(\mathbb{R}^n)$ , where

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(\mathbb{R}^n) = \left\{ f \in L_{\text{loc}}^{q(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \sum_{k=-\infty}^{\infty} \|2^{k\alpha(\cdot)} f \chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p < \infty \right\}.$$

Obviously,  $M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),0}(\mathbb{R}^n) = \dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(\mathbb{R}^n)$ . When  $\alpha(\cdot)$  is constant, we have  $M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n) = M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$  (see [9]). If both  $\alpha(\cdot)$  and  $q(\cdot)$  are constant, and  $\lambda = 0$ , then  $M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$  are classical Herz spaces.

## 2.2 Auxiliary propositions and lemmas

In this part we state some auxiliary propositions and lemmas which will be needed for proving our main theorems. And we only describe partial results we need.

**Proposition 2.1** Let  $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ .

- (I) If  $q(\cdot) \in \mathcal{C}^{\log}(\mathbb{R}^n)$ , then we have  $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ .
- (II)  $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$  if and only if  $q'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ .

The first part in Proposition 2.1 is independently due to Cruz-Uribe et al<sup>[18]</sup> and to Nekvinda<sup>[24]</sup> respectively. The second of Proposition 2.1 belongs to Diening<sup>[19]</sup> (see Theorem 8.1 or Theorem 1.2 in [17]).

**Remark 2** Since

$$|q'(x) - q'(y)| \leq \frac{|q(x) - q(y)|}{(q_- - 1)^2},$$

it follows at once that if  $q(\cdot) \in \mathcal{C}^{\log}(\mathbb{R}^n)$ , then so does  $q'(\cdot)$ —i.e., if the condition hold, then  $M$  is bounded on  $L^{q(\cdot)}(\mathbb{R}^n)$  and  $L^{q'(\cdot)}(\mathbb{R}^n)$ . Furthermore, Diening has proved general results on Musielak-Orlicz spaces.

The order  $\beta(x)$  of the fractional Hardy-type operators in Definition 1.1 is not assumed to be continuous. We assume that it is a measurable function on  $\mathbb{R}^n$  satisfying the following assumptions

$$\left. \begin{aligned} \beta_0 &:= \operatorname{ess\,inf}_{x \in \mathbb{R}^n} \beta(x) > 0 \\ \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x)\beta(x) &< n \\ \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(\infty)\beta(x) &< n \end{aligned} \right\}. \quad (2.3)$$

In order to prove our main results, we need the Sobolev type theorem for the space  $\mathbb{R}^n$  which was proved in ref. [28] for the exponents  $p(x)$  not necessarily constant in a neighbourhood of infinity, but with some extra power weight fixed to infinity and under the assumption that  $p(x)$  takes its minimal value at infinity.

**Proposition 2.2** Suppose that  $p(\cdot) \in \mathcal{C}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ . Let

$$1 < p(\infty) \leq p(x) \leq p_+ < \infty, \quad (2.4)$$

and  $\beta(x)$  meet condition (2.3). Then the following weighted Sobolev-type estimate is valid for the operator  $I_{\beta(\cdot)}$ :

$$\left\| (1 + |x|)^{-\gamma(x)} I_{\beta(\cdot)}(f) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\beta(x)}{n}$$

is the Sobolev exponent and

$$\gamma(x) = C_\infty \beta(x) \left( 1 - \frac{\beta(x)}{n} \right) \leq \frac{n}{4} C_\infty, \quad (2.5)$$

$C_\infty$  being the Dini-Lipschitz constant from (2.2) which  $q(\cdot)$  is replaced by  $p(\cdot)$ .

**Remark 3** (i) If  $\beta(x)$  satisfies the condition of type (2.2):  $|\beta(x) - \beta_\infty| \leq \frac{C_\infty}{\ln(e+|x|)}$  ( $x \in \mathbb{R}^n$ ), then the weight  $(1 + |x|)^{-\gamma(x)}$  is equivalent to the weight  $(1 + |x|)^{-\gamma_\infty}$ .

(ii) One can also treat operator (1.1) with  $\beta(x)$  replaced by  $\beta(y)$ . In the case of potentials over bounded domains  $\Omega$  such potentials differ unessentially, if the function  $\beta(x)$  satisfies the smoothness logarithmic condition as (2.1), since

$$C_1 |x - y|^{n-\beta(y)} \leq |x - y|^{n-\beta(x)} \leq C_2 |x - y|^{n-\beta(y)}$$

in this case ( see [27], p. 277).

(iii) When  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ , the assumption that  $p(\cdot) \in \mathcal{C}^{\log}(\mathbb{R}^n)$  is equivalent to assuming  $1/p(\cdot) \in \mathcal{C}^{\log}(\mathbb{R}^n)$ , since

$$\left| \frac{p(x) - p(y)}{(p_+)^2} \right| \leq \left| \frac{1}{p(x)} - \frac{1}{p(y)} \right| = \left| \frac{p(x) - p(y)}{p(x)p(y)} \right| \leq \left| \frac{p(x) - p(y)}{(p_-)^2} \right|.$$

And further,  $1/p(\cdot) \in \mathcal{C}^{\log}(\mathbb{R}^n)$  implies that  $1/q(\cdot) \in \mathcal{C}^{\log}(\mathbb{R}^n)$  as well.

The next proposition is the generalization of variable exponents Herz spaces in [29], and it was used in [34].

**Proposition 2.3** Let  $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ ,  $p \in (0, \infty)$ , and  $\lambda \in [0, \infty)$ . If  $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{C}_0^{\log}(\mathbb{R}^n) \cap \mathcal{C}_\infty^{\log}(\mathbb{R}^n)$ , then

$$\begin{aligned} \|f\|_{M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)} &= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left( \sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f \chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right)^{\frac{1}{p}} \\ &\approx \max \left\{ \sup_{\substack{k_0 < 0 \\ k_0 \in \mathbb{Z}}} 2^{-k_0\lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p} \|f \chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}, \right. \\ &\quad \sup_{\substack{k_0 \geq 0 \\ k_0 \in \mathbb{Z}}} \left( 2^{-k_0\lambda} \left( \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p} \|f \chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right)^{\frac{1}{p}} \right. \\ &\quad \left. \left. + 2^{-k_0\lambda} \left( \sum_{k=0}^{k_0} 2^{k\alpha_\infty p} \|f \chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right)^{\frac{1}{p}} \right) \right\}. \end{aligned}$$

The next lemma is known as the generalized Hölder's inequality on Lebesgue spaces with variable exponent, and the proof can be found in [15].

**Lemma 2.1** (generalized Hölder's inequality) Suppose that  $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ , then for any  $f \in L^{q(\cdot)}(\mathbb{R}^n)$  and any  $g \in L^{q'(\cdot)}(\mathbb{R}^n)$ , we have

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq C_q \|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{q'(\cdot)}(\mathbb{R}^n)},$$

where  $C_q = 1 + 1/q_- - 1/q_+$ .

The following lemma can be found in [35].

**Lemma 2.2** Let  $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ .

(I) Then there exist positive constants  $\delta \in (0, 1)$  and  $C > 0$  such that

$$\frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \leq C \left( \frac{|S|}{|B|} \right)^\delta$$

for all balls  $B$  in  $\mathbb{R}^n$  and all measurable subsets  $S \subset B$ .

(II) Then there exists a positive constant  $C > 0$  such that

$$C^{-1} \leq \frac{1}{|B|} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \leq C$$

for all balls  $B$  in  $\mathbb{R}^n$ .

**Remark 4** (i) If  $q_1(\cdot), q_2(\cdot) \in \mathcal{C}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ , then we see that  $q'_1(\cdot), q_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ . Hence we can take positive constants  $0 < \delta_1 < 1/(q'_1)_+$ ,  $0 < \delta_2 < 1/(q_2)_+$  such that

$$\frac{\|\chi_S\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)}} \leq C \left( \frac{|S|}{|B|} \right)^{\delta_1}, \quad \frac{\|\chi_S\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}} \leq C \left( \frac{|S|}{|B|} \right)^{\delta_2} \quad (2.6)$$

hold for all balls  $B$  in  $\mathbb{R}^n$  and all measurable subsets  $S \subset B$  ( see [6, 35]).

(ii) On the other hand, Kopaliani<sup>[22]</sup> has proved the conclusion: If the exponent  $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  equals to a constant outside some large ball, then  $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$  if and only if  $q(\cdot)$  satisfies the Muckenhoupt type condition

$$\sup_{Q:\text{cube}} \frac{1}{|Q|} \|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)} < \infty.$$

### 3 Main results and their proofs

Our main result can be stated as follows (some details see [9]).

**Theorem 3.1** Suppose that  $q_1(\cdot) \in \mathcal{C}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$  satisfies condition (2.4), and  $\beta(x)$  meet condition (2.3) which  $p(\cdot)$  is replaced by  $q_1(\cdot)$ . Define the variable exponent  $q_2(\cdot)$  by

$$\frac{1}{q_2(x)} = \frac{1}{q_1(x)} - \frac{\beta(x)}{n}.$$

Let  $0 < p_1 \leq p_2 < \infty$ ,  $\lambda \geq 0$ , and  $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$  be log-Hölder continuous both at the origin and at infinity, with  $\alpha(0) \leq \alpha_\infty < \lambda + n\delta_1$ , where  $\delta_1 \in (0, 1/(q'_1)_+)$  is the constant appearing in (2.6). Then

$$\left\| (1 + |x|)^{-\gamma(x)} \mathcal{H}_{\beta(\cdot)}(f) \right\|_{M\dot{K}_{p_2, q_2(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)} \leq C \|f\|_{M\dot{K}_{p_1, q_1(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)},$$

where  $\gamma(x)$  is defined as in (2.5), and  $C_\infty$  is the Dini-Lipschitz constant from (2.1) which  $q_1(\cdot)$  instead of  $q(\cdot)$ .

**Proof** For any  $f \in M\dot{K}_{p_1, q_1(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)$ , if we denote  $f_j := f \cdot \chi_j = f \cdot \chi_{A_j}$  for each  $j \in \mathbb{Z}$ , then we can write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x) \cdot \chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

By (1.0a) and Lemma 2.1, we have

$$\begin{aligned} |\mathcal{H}_{\beta(\cdot)}(f)(x) \cdot \chi_k(x)| &\leq \frac{1}{|x|^{n-\beta(x)}} \int_{B_k} |f(t)| dt \cdot \chi_k(x) \\ &\leq C 2^{-kn} \sum_{j=-\infty}^k \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \cdot |x|^{\beta(x)} \chi_k(x). \end{aligned} \quad (3.1)$$

For Proposition 2.2, we note that

$$\begin{aligned} I_{\beta(\cdot)}(\chi_{B_k})(x) &\geq I_{\beta(\cdot)}(\chi_{B_k})(x) \cdot \chi_{B_k}(x) = \int_{B_k} \frac{1}{|x-y|^{n-\beta(x)}} dy \cdot \chi_{B_k}(x) \\ &\geq C |x|^{\beta(x)} \cdot \chi_{B_k}(x) \geq C |x|^{\beta(x)} \cdot \chi_k(x). \end{aligned} \quad (3.2)$$

Using Proposition 2.2, Lemma 2.2, (2.6), (3.1) and (3.2), we have

$$\begin{aligned} &\left\| (1 + |x|)^{-\gamma(x)} \mathcal{H}_{\beta(\cdot)}(f) \cdot \chi_k(\cdot) \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C 2^{-kn} \sum_{j=-\infty}^k \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \left\| (1 + |x|)^{-\gamma(x)} I_{\beta(\cdot)}(\chi_{B_k}) \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C 2^{-kn} \sum_{j=-\infty}^k \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\ &\leq C \sum_{j=-\infty}^k \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)}} \leq C \sum_{j=-\infty}^k 2^{(j-k)n\delta_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}. \end{aligned} \quad (3.3)$$

Because of  $0 < p_1/p_2 \leq 1$ , applying inequality

$$\left( \sum_{i=-\infty}^{\infty} |a_i| \right)^{p_1/p_2} \leq \sum_{i=-\infty}^{\infty} |a_i|^{p_1/p_2}, \quad (3.4)$$



and Proposition 2.3, then we have

$$\begin{aligned}
& \left\| (1 + |x|)^{-\gamma(x)} \mathcal{H}_{\beta(\cdot)}(f) \right\|_{\dot{M}_{p_2, q_2(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)}^{p_1} \\
& \leq \max \left\{ \sup_{\substack{k_0 < 0 \\ k_0 \in \mathbb{Z}}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha(0) p_1} \left\| (1 + |x|)^{-\gamma(x)} \mathcal{H}_{\beta(\cdot)}(f) \cdot \chi_k \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{p_1} \right), \right. \\
& \quad \sup_{\substack{k_0 \geq 0 \\ k_0 \in \mathbb{Z}}} \left( 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{-1} 2^{k \alpha(0) p_1} \left\| (1 + |x|)^{-\gamma(x)} \mathcal{H}_{\beta(\cdot)}(f) \cdot \chi_k \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{p_1} \right) \right. \\
& \quad \left. \left. + 2^{-k_0 \lambda p_1} \left( \sum_{k=0}^{k_0} 2^{k \alpha_\infty p_1} \left\| (1 + |x|)^{-\gamma(x)} \mathcal{H}_{\beta(\cdot)}(f) \cdot \chi_k \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{p_1} \right) \right) \right\} \\
& = \max\{E_1, E_2 + E_3\},
\end{aligned}$$

where

$$\begin{aligned}
E_1 &= \sup_{\substack{k_0 < 0 \\ k_0 \in \mathbb{Z}}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha(0) p_1} \left\| (1 + |x|)^{-\gamma(x)} \mathcal{H}_{\beta(\cdot)}(f) \cdot \chi_k \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{p_1} \right), \\
E_2 &= \sup_{\substack{k_0 \geq 0 \\ k_0 \in \mathbb{Z}}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{-1} 2^{k \alpha(0) p_1} \left\| (1 + |x|)^{-\gamma(x)} \mathcal{H}_{\beta(\cdot)}(f) \cdot \chi_k \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{p_1} \right), \\
E_3 &= \sup_{\substack{k_0 \geq 0 \\ k_0 \in \mathbb{Z}}} 2^{-k_0 \lambda p_1} \left( \sum_{k=0}^{k_0} 2^{k \alpha_\infty p_1} \left\| (1 + |x|)^{-\gamma(x)} \mathcal{H}_{\beta(\cdot)}(f) \cdot \chi_k \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{p_1} \right).
\end{aligned}$$

To estimate  $E_1, E_2$  and  $E_3$ , we need the following fact. By the condition of  $\alpha(\cdot)$  and Proposition 2.3, we have

Case 1 ( $j < 0$ ),

$$\begin{aligned}
\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} &= 2^{-j \alpha(0)} \left( 2^{j \alpha(0) p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \right)^{1/p_1} \\
&\leq 2^{-j \alpha(0)} \left( \sum_{i=-\infty}^j 2^{i \alpha(0) p_1} \|f_i\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \right)^{1/p_1} \\
&\leq 2^{j(\lambda - \alpha(0))} \left( 2^{-j \lambda} \left( \sum_{i=-\infty}^j \|2^{i \alpha(\cdot)} f_i\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \right)^{1/p_1} \right) \\
&\leq C 2^{j(\lambda - \alpha(0))} \|f\|_{\dot{M}_{p_1, q_1(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)}.
\end{aligned} \tag{3.5}$$

Case 2 ( $j \geq 0$ ),

$$\begin{aligned}
\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} &= 2^{-j\alpha_\infty} \left( 2^{j\alpha_\infty p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \right)^{1/p_1} \\
&\leq 2^{-j\alpha_\infty} \left( \sum_{i=0}^j 2^{i\alpha_\infty p_1} \|f_i\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \right)^{1/p_1} \\
&\leq 2^{j(\lambda-\alpha_\infty)} \left( 2^{-j\lambda} \left( \sum_{i=-\infty}^j \|2^{i\alpha(\cdot)} f_i\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \right)^{1/p_1} \right) \\
&\leq C 2^{j(\lambda-\alpha_\infty)} \|f\|_{M\dot{K}_{p_1, q_1(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)}.
\end{aligned} \tag{3.6}$$

For  $E_1$ , note that  $j < 0$ , combining (3.3) and (3.5), and using  $\alpha(0) \leq \alpha_\infty < \lambda + n\delta_1$ , it follows that

$$\begin{aligned}
E_1 &\leq C \sup_{\substack{k_0 < 0 \\ k_0 \in \mathbb{Z}}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1} \left( \sum_{j=-\infty}^k 2^{(j-k)n\delta_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \right) \\
&\leq C \|f\|_{M\dot{K}_{p_1, q_1(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)}^{p_1} \sup_{\substack{k_0 < 0 \\ k_0 \in \mathbb{Z}}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1} \left( \sum_{j=-\infty}^k 2^{(j-k)n\delta_1} 2^{j(\lambda-\alpha(0))} \right)^{p_1} \right) \\
&\leq C \|f\|_{M\dot{K}_{p_1, q_1(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)}^{p_1} \sup_{\substack{k_0 < 0 \\ k_0 \in \mathbb{Z}}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k\lambda p_1} \left( \sum_{j=-\infty}^k 2^{(j-k)(n\delta_1 + \lambda - \alpha(0))} \right)^{p_1} \right) \\
&\leq C \|f\|_{M\dot{K}_{p_1, q_1(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)}^{p_1} \sup_{\substack{k_0 < 0 \\ k_0 \in \mathbb{Z}}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k\lambda p_1} \right) \leq C \|f\|_{M\dot{K}_{p_1, q_1(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)}^{p_1}.
\end{aligned}$$

We omit the estimate of  $E_2$  since it is essentially similar to that of  $E_1$ .

Now we only simply estimate  $E_3$ , note that  $j \geq 0$ , combining (3.3) and (3.6), and using  $\alpha(0) \leq \alpha_\infty < \lambda + n\delta_1$ , we have

$$\begin{aligned}
E_3 &\leq C \sup_{\substack{k_0 \geq 0 \\ k_0 \in \mathbb{Z}}} 2^{-k_0 \lambda p_1} \left( \sum_{k=0}^{k_0} 2^{k\alpha_\infty p_1} \left( \sum_{j=-\infty}^k 2^{(j-k)n\delta_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \right) \\
&\leq C \|f\|_{M\dot{K}_{p_1, q_1(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)}^{p_1} \sup_{\substack{k_0 \geq 0 \\ k_0 \in \mathbb{Z}}} 2^{-k_0 \lambda p_1} \left( \sum_{k=0}^{k_0} 2^{k\lambda p_1} \left( \sum_{j=-\infty}^k 2^{(j-k)(n\delta_1 + \lambda - \alpha_\infty)} \right)^{p_1} \right) \\
&\leq C \|f\|_{M\dot{K}_{p_1, q_1(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)}^{p_1} \sup_{\substack{k_0 \geq 0 \\ k_0 \in \mathbb{Z}}} 2^{-k_0 \lambda p_1} \left( \sum_{k=0}^{k_0} 2^{k\lambda p_1} \right) \leq C \|f\|_{M\dot{K}_{p_1, q_1(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)}^{p_1}.
\end{aligned}$$

Combining all the estimates for  $E_i$  ( $i = 1, 2, 3$ ) together, the proof of Theorem 3.1 is completed.  $\blacksquare$

**Theorem 3.2** Let  $\lambda, p_1, p_2, q_1(\cdot), q_2(\cdot), \beta(x), C_\infty$  be as in Theorem 3.1. Suppose that  $\alpha \in L^\infty(\mathbb{R}^n)$  be log-Hölder continuous both at the origin and at infinity, and  $\lambda - n\delta_2 < \alpha(0) \leq \alpha_\infty$ , where  $\delta_2 \in (0, 1/(q_2)_+)$  is the constant appearing in (2.6). Then

$$\left\| (1 + |x|)^{-\gamma(x)} \mathcal{H}_{\beta(\cdot)}^*(f) \right\|_{M\dot{K}_{p_2, q_2(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)} \leq C \|f\|_{M\dot{K}_{p_1, q_1(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)}.$$

**Proof** Similar to the proof of Theorem 3.1, therefore, we only give a simple proof. For simplicity, for any  $f \in M\dot{K}_{p_1, q_1(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)$ , we write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x) \cdot \chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

By (1.0b) and Lemma 2.1, we have

$$\begin{aligned} \left| (1 + |x|)^{-\gamma(x)} \mathcal{H}_{\beta(\cdot)}^*(f)(x) \cdot \chi_k(x) \right| &\leq C \int_{\mathbb{R}^n \setminus B_k} |f(t)| |x|^{\beta(x)-n} dt \cdot (1 + |x|)^{-\gamma(x)} \chi_k(x) \\ &\leq C \sum_{j=k+1}^{\infty} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \left\| (1 + |x|)^{-\gamma(x)} \cdot |x|^{\beta(x)-n} \chi_j(\cdot) \right\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \cdot \chi_k(x). \end{aligned} \quad (3.7)$$

Similar to (3.2), we give

$$I_{\beta(\cdot)}(\chi_{B_j})(x) \geq I_{\beta(\cdot)}(\chi_{B_j})(x) \cdot \chi_{B_j}(x) \geq C|x|^{\beta(x)} \cdot \chi_j(x). \quad (3.8)$$

Applying Proposition 2.2, Lemma 2.2, (2.6), (3.7) and (3.8), we obtain

$$\begin{aligned} &\left\| (1 + |x|)^{-\gamma(x)} \mathcal{H}_{\beta(\cdot)}^*(f) \cdot \chi_k(\cdot) \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C \sum_{j=k+1}^{\infty} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \cdot 2^{-jn} \left\| (1 + |x|)^{-\gamma(x)} I_{\beta(\cdot)}(\chi_{B_j}) \right\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \\ &\leq C \sum_{j=k+1}^{\infty} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}} \leq C \sum_{j=k+1}^{\infty} 2^{(k-j)n\delta_2} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}. \end{aligned} \quad (3.9)$$

By (3.4) and Proposition 2.3, we have

$$\left\| (1 + |x|)^{-\gamma(x)} \mathcal{H}_{\beta(\cdot)}^*(f) \right\|_{M\dot{K}_{p_2, q_2(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)}^{p_1} \leq \max\{E_1, E_2 + E_3\},$$

where

$$\begin{aligned} E_1 &= \sup_{\substack{k_0 < 0 \\ k_0 \in \mathbb{Z}}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1} \left\| (1 + |x|)^{-\gamma(x)} \mathcal{H}_{\beta(\cdot)}^*(f) \cdot \chi_k \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{p_1} \right), \\ E_2 &= \sup_{\substack{k_0 \geq 0 \\ k_0 \in \mathbb{Z}}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p_1} \left\| (1 + |x|)^{-\gamma(x)} \mathcal{H}_{\beta(\cdot)}^*(f) \cdot \chi_k \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{p_1} \right), \\ E_3 &= \sup_{\substack{k_0 \geq 0 \\ k_0 \in \mathbb{Z}}} 2^{-k_0 \lambda p_1} \left( \sum_{k=0}^{k_0} 2^{k\alpha_{\infty} p_1} \left\| (1 + |x|)^{-\gamma(x)} \mathcal{H}_{\beta(\cdot)}^*(f) \cdot \chi_k \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{p_1} \right). \end{aligned}$$

For  $E_1, E_2$  and  $E_3$ , combining (3.5), (3.6) and (3.9), and using  $\lambda - n\delta_2 < \alpha(0) \leq \alpha_{\infty}$ , we have

$$E_i \leq C \|f\|_{M\dot{K}_{p_1, q_1(\cdot)}^{\alpha(\cdot), \lambda}(\mathbb{R}^n)}^{p_1}, \quad (i = 1, 2, 3).$$

Combining all the estimates for  $E_i$  ( $i = 1, 2, 3$ ) together, the proof of Theorem 3.2 is completed.  $\blacksquare$

In particular, when  $\gamma(x) = 0$ ,  $\alpha(\cdot)$  and  $\beta(\cdot)$  are constant exponent, the main results above are proved by Zhang and Wu in [8]. Let  $\alpha(\cdot)$  be constant exponent, then the above results can be founded in [9]. And when  $\lambda = 0$ , our main results are also valid.

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